TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 2 2018

Section 1 Calculator Free
Vectors

STUDENT'S NAME SOLUTIONS

DATE: Thursday 29 March **TIME:** 20 minutes **MARKS**: 22

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

The points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$; $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 7 \\ -3 \\ 10 \end{pmatrix}$ respectively. Show that angle ABC is a right angle.

$$\vec{AB} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} 7 \\ -3 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix}$$

$$AB \cdot BC = -15 + 8 + 7 = 0$$

$\therefore$ RT ANGLE

2. (5 marks)

(a) A sphere has Cartesian equation $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$. Determine the coordinates of the centre and the radius of the sphere. (4 marks)

$$x^2 - 2x + 1 \quad + y^2 + 4y + 4 + z^2 - 6z + 9 = 11 + 1 + 4 + 9$$

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 25$$

CENTRE $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

RAD 5

(b) Write the vector equation of the sphere from part (a) (1 mark)

$$\left| r - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right| = 5$$

3. (8 marks)

The position vectors of the points A and B are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ respectively.

(a) Determine the vector equation of the line joining A and B. (2 marks)

$$AB = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

(b) Determine the cartesian equation of the line joining A and B. (3 marks)

$x = 2 - \lambda$ $\qquad$ $\lambda = 2 - x$

$y = -3 + 8\lambda$

$$y = -3 + 8(2 - x)$$

$$= -3 + 16 - 8x$$

$$= 13 - 8x$$

(c) Determine the cosine of the angle the line AB makes with the x-axis. (3 marks)

x AXIS $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ LINE $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$

$$x \cdot r = |x||r| \cos\theta$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 \\ 8 \end{pmatrix} = 1 \times \sqrt{65} \cos\theta$$

$$\frac{-1}{\sqrt{65}} = \cos\theta$$

4. (5 marks)

Determine a unit vector perpendicular to the plane that contains the points with position vectors $A\begin{pmatrix}1\\2\\1\end{pmatrix}$, $B\begin{pmatrix}-2\\1\\3\end{pmatrix}$ and $C\begin{pmatrix}0\\2\\4\end{pmatrix}$.

$AB = \begin{pmatrix}-3\\-1\\2\end{pmatrix}$

$BC = \begin{pmatrix}2\\1\\1\end{pmatrix}$

$AB \times BC = i(-1-2) - j(-3-4) + k(-3+2)$

$n = -3i + 7j - k$

$|n| = \sqrt{9+49+1}$

$\hat{n} = \dfrac{-3i + 7j - k}{\sqrt{59}}$

TRINITY COLLEGE

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Mathematics Specialist Units 3,4
Test 2 2018

Section 2 Calculator Assumed

Vectors

STUDENT'S NAME ______________________________

DATE: Thursday 29 March **TIME:** 30 minutes **MARKS:** 32

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (5 marks)

Two particles A and B have initial position vectors $-4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ respectively and their respective constant velocity vectors are $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. (Units are metres and seconds)

(a) Determine the position vector of where the paths of the particles cross. (3 marks)

$r_A = \begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $r_B = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$4 + 2\lambda = 3 + \mu$

$2 + \lambda = 4 + 2\mu$

ie $\lambda = 4$

$\mu = 1$

CROSS AT $\begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$

(b) Do the particles meet? Explain. (2 marks)

NO SINCE $\lambda \neq \mu$ (OR $T_1 \neq T_2$)

PARTICLES DO NOT MEET

6. (5 marks)

The position vector of a particle is given by $\mathbf{r} = \begin{pmatrix} 2 + 4\sin 2t \\ 4\cos 2t - 3 \end{pmatrix}$, where t is time in seconds.

(a) State the initial position vector of the particle. (1 mark)

$t = 0$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) Determine the Cartesian equation of the path of the particle. (4 marks)

$x = 2 + 4\sin 2t$ $\sin 2t = \frac{x-2}{4}$

$y = 4\cos 2t - 3$ $\cos 2t = \frac{y+3}{4}$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y+3}{4}\right)^2 = 1$

$(x-2)^2 + (y+3)^2 = 16$

7. (5 marks)

Determine the minimum distance between the point with position vector $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ and the plane $2x-4y+z=1$.

PLANE $\quad r \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 1$

POINT A $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

POINT B $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (ANY POINT ON THE PLANE)

$$AB = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

$$r = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3+2\lambda \\ -4\lambda \\ -2+\lambda \end{pmatrix}$$

$$\begin{pmatrix} 3+2\lambda \\ -4\lambda \\ -2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 1$$

$$4 + 21\lambda = 1$$

$$\lambda = -\frac{1}{7}$$

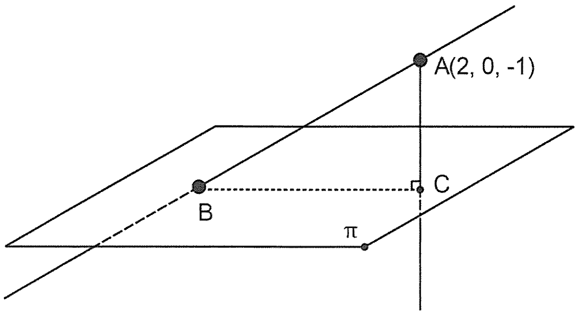
INTERSECTION AT $\begin{pmatrix} \frac{19}{7} \\ \frac{4}{7} \\ -\frac{15}{7} \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{19}{7} \\ \frac{4}{7} \\ -\frac{15}{7} \end{pmatrix} = \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ \frac{1}{7} \end{pmatrix}$$

$$\left| \begin{matrix} \frac{2}{7} \\ -\frac{4}{7} \\ \frac{1}{7} \end{matrix} \right| = \frac{\sqrt{21}}{7}$$

8. (8 marks)

The point A has coordinates (2, 0, −1) and the plane Π has the equation $x + 2y - 2z = 8$. The line through A parallel to the line $\frac{x}{-2} = y = \frac{z+1}{2}$ meets Π at B and the perpendicular from A to Π meets Π at the point C as shown in the diagram.



(a) Determine the equations of the two lines AB and AC. (4 marks)

$\frac{x}{-2} = y = \frac{z+1}{2}$

$y = \lambda$
$x = -2\lambda$
$z = 2\lambda - 1$

$\left.\begin{matrix} y = \lambda \\ x = -2\lambda \\ z = 2\lambda - 1 \end{matrix}\right\}$ DIRECTION $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$r_{AB} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$r_{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

(b) Determine the coordinates of the points B and C. (4 marks)

$\begin{pmatrix} 2-2\lambda \\ \lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 8$

$\lambda = -1$

B $\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$

$r \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 8$

$\begin{pmatrix} 2+\lambda \\ 2\lambda \\ -1-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 8$

$\lambda = \frac{4}{9}$

C $\begin{pmatrix} \frac{22}{9} \\ \frac{8}{9} \\ \frac{-17}{9} \end{pmatrix}$

9. (9 marks)

A plane Π contains two lines $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{j} + 7\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(-5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$.

(a) Write down a vector equation of the plane Π in the form $\mathbf{r} = \mathbf{a} + \alpha\mathbf{b} + \beta\mathbf{c}$. (1 mark)

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} + \mu\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

(b) The point $-9\mathbf{i} + 2\mathbf{j} + n\mathbf{k}$ lies in the plane Π. Determine the value of the constant n. (3 marks)

$$\begin{pmatrix} -9 \\ 2 \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} + \mu\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

$-9 = 1 - 5\mu$ $\qquad\qquad$ $\lambda = -5$

$2 = -1 + \lambda + 4\mu$ $\qquad$ $\mu = 2$

$n = -27$

(c) The vector $m\begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$ is normal to the plane. Determine the values of the constants m and b. (3 marks)

$$\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} \times \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = m\begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$$

$$\begin{pmatrix} -25 \\ -35 \\ 5 \end{pmatrix} = m\begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$$

$$-5\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = m\begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$$

$m = -5$

$b = -1$

(d) State the equation of the plane Π in the form $\mathbf{r} \cdot \mathbf{n} = c$. (2 marks)

$$r \cdot \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = 20$$